Example Compute

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

or show it does not exist.

Solution. Let's try along some easy paths, like $y=mx$ (m some constant)

$$\lim_{x\to 0} \frac{x^3+m^3x^3}{x^2+m^2x^2} = \lim_{x\to 0} x\,\frac{1+m^3}{1+m^2}$$

$$= 0$$

This does not let us conclude the limit is 0. (Unless we knew in advance that the limit exists)

Seeing r^2 in denom suggests switching to polar:

$$\lim_{\substack{r\to 0^+ \\ \theta \text{ any}}} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2}$$

$$= \lim_{\substack{r\to 0^+ \\ \theta \text{ any}}} r\left(\cos^3\theta + \sin^3\theta\right)$$

Note:

$$-2r \le r\left(\cos^3\theta + \sin^3\theta\right) \le 2r$$

these $\to 0$ for $\substack{r\to 0^+ \\ \theta \text{ any}}$

so by the Squeeze Thm

$$\lim_{\substack{r\to 0^+ \\ \theta \text{ any}}} r\left(\cos^3\theta + \sin^3\theta\right) = 0.$$

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^3}{x^2 + 2y^2}$$

$$v = \sqrt{2}\, y$$

$$= \lim_{(x,v)\to(0,0)} \frac{x^4 - \frac{4}{2\sqrt{2}} v^3}{x^2 + v^2}$$

$$x = r\cos\theta \qquad v = r\sin\theta$$

$$= \lim_{\substack{r\to 0^+ \\ \theta \text{ any}}} \frac{}{r^2}$$

etc.